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Interpolation of functions from generalized Paley–Wiener spaces

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Abstract

We consider the problem of reconstruction of functions f from generalized Paley–Wiener spaces in terms of their values on complete interpolating sequence $\{z_n\}$. We characterize the set of data sequences $\{f(z_n)\}$ and exhibit an explicit solution to the problem. Our development involves the solution of a particular $\bar{\partial}$ problem.

Keywords: Paley—Wiener spaces; Complete interpolating sequences; $\bar{\partial}$ problem; Divided differences

1. Introduction

In this article, we consider questions of interpolation in the generalized Paley–Wiener space $PW^{p,k}$ consisting of entire functions F of exponential-type π whose derivative of order k, $F^{(k)}$, belongs to $L^p(\mathbb{R})$ when restricted to the real axis. The space $PW^{p,k}$ is naturally endowed with the seminorm

$$\|F\|_{PW^{p,k}} = \|F^{(k)}\|_{L^p(\mathbb{R})}.$$

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In our considerations the parameter p satisfies the restriction $1 < p < \infty$ and k is a non-negative integer.

Before describing the contents more explicitly we briefly review some background material.

If $k = 0$ then $\|F\|_{PW^{p,k}}$ is a norm while in the case $k > 0$ it is a seminorm whose null space is the class of polynomials of degree $\leqslant k-1$. Also note that $PW^{p,k} \subset PW^{p,k+1}$ as follows from the appropriate version of the Bernstein inequality. Below we write just PW^p for $PW^{p,0}$.

In the engineering literature the classical Paley–Wiener space PW^2 is viewed as the class of frequency band limited functions of finite power.

The classical Whittaker–Kotel'nikov–Shannon sampling theorem characterizes the data samples $\{F(n)\}_{n\in\mathbb{Z}}$ of elements F in PW^2 sampled on the integer lattice $\mathbb{Z}$ and provides an explicit reconstruction formula,

$$F(z) = \sum_{n=-\infty}^{\infty} F(n) \frac{\sin \pi(z-n)}{\pi(z-n)} \tag{1}$$

for F in terms of the data samples. For details see, for example, [5], [11] or [20]. Formula (1) is commonly known as the Whittaker cardinal sine series.

This result has been extended in several directions.

(i) The sequences $\mathcal{Z} = \{z_n\}$ which ensure that the mapping of F to the samples $\{F(z_n)\}_{n\in\mathbb{Z}}$ is an isomorphism between PW^p and l^p have been completely characterized. The inverse of this mapping is given by the explicit reconstruction formula

$$F(z) = \sum_{z_n\in\mathcal{Z}} F(z_n) \frac{S(z)}{S'(z_n)(z-z_n)}. \tag{2}$$

Here S is the generating function of $\mathcal{Z}$. It is defined by the relation

$$S(z) = \lim_{r\to\infty} (z-z_0) \prod_{\{z_n\in\mathcal{Z}:|z_n|<r, n\neq 0\}} \left(1 - \frac{z}{z_n}\right), \tag{3}$$

we assume that $z_n \neq 0$ for $n \neq 0$.

Such sequences $\mathcal{Z}$ are called complete interpolating sequences for PW^p. Results in this direction go back to the 1936 work of Paley and Wiener [15], represent the contribution of many investigators, and are summarized in [7,13,20]. We also refer the reader to Section 2 for more details. Everywhere below we assume that $\mathcal{Z}$ is a complete interpolating sequence for $PW^{p,k}$.

(ii) In 1974, Schoenberg [17] characterized the data samples $\{F(n)\}_{n\in\mathbb{Z}}$ of entire functions in $PW^{2,k}$ and showed that splines could be used in a summability-type reconstruction procedure. Since the class of such entire functions is larger than PW^2, for instance it contains polynomials of degree less than k as well as other functions which fail to be square integrable on the real axis, this result extends the classical sampling theorem in a certain sense.

(iii) The results described in (ii) have been extended to cases where $\mathbb{Z}$ is replaced by a more general interpolating sequences $\mathcal{Z}$. Specifically, in [14] the data samples $\{F(z_n)\}$ of entire functions $F \in PW^{2,k}$ are characterized in cases when the sequences of sampling nodes $\{z_n\}$ are real complete interpolating sequences for PW^2 and several summability-type reconstruction procedures for F in terms of such data samples are provided.

In this article, we develop another approach which leads one to explicit reconstruction formulas analogous to (1) and (2). These formulas are new even in the classical case $\mathcal{Z} = \mathbb{Z}$. This also allows us to extend the results mentioned in item (iii) to include the spaces $PW^{p,k}$, $1 < p < \infty$, and remove the restriction that the sampling sequence $\mathcal{Z}$ be real.

Before going into details we bring attention to several issues which make the problem under consideration of interest to us.

For $k > 0$ the class $PW^{p,k}$ is considerably wider than PW^p. For example, it contains polynomials of degree $\leqslant k-1$ as well as many other entire functions which fail to decay on the real axis. Nevertheless, one can prove (see Section 3.2 for the details) that each PW^p complete interpolating sequence still is a set of uniqueness for the spaces $PW^{p,k}$. So the question appears about how one can reconstruct functions from these wider spaces in terms of their samples at a PW^p-complete interpolating sequence.

We deal with this matter by considering a more complicated interpolation problem. Namely, instead of simple pointwise interpolation we consider the interpolation of certain linear combinations, divided differences, of point values. This presents an interesting challenge because the routine interpolation tools such as Lagrange-type interpolation series cannot be applied directly for such a non-local problem. As a consequence, we were led to develop techniques of interpolation associated with the $\bar{\partial}$ problem and combine them with the classical interpolation techniques (see e.g. [11]) for Paley–Wiener spaces. We refer the reader to [6], Chapter IV for an account of the $\bar{\partial}$ problem.

The remainder of this article is organized as follows: Sections 2 contains the precise statements of the main results including the necessary background material. All the technical details are relegated to Section 3.

2. Preliminaries and the main results

2.1. Complete interpolating sequences and generating functions

We start by describing the basic properties of complete interpolating sequences. We refer the reader to [13] for more details and further references.

Recall that a sequence $\mathcal{Z} = \{z_k\} \subset \mathbb{C}$, $z_k = x_k + iy_k$ is called a complete interpolating sequence for PW^p if for each sequence $\{a_k\} \in l^p$ the interpolation problem

$$F(z_k) = a_k$$

has a unique solution $F \in PW^p$.

Theorem A (*Lyubarskii and Seip [13] and Pavlov [16]*). *In order that a sequence $\mathcal{Z} = \{z_k\}_{-\infty}^{\infty} \subset \mathbb{C}$ be a complete interpolating sequence for PW^p it is necessary and sufficient that*

- $H := \sup |y_k| < \infty$,

- *$\mathcal{Z}$ is uniformly discrete, i.e.*

$$\inf_{k\neq l} |z_k - z_l| > 0. \tag{4}$$

- *The canonical product* (3) *converges on each compact set in $\mathbb{C}$ to an entire function S of exponential-type π and for each real a such that $|a| > 2H$ the function $w(x) = |S(x+ia)|^p$ satisfies the $(\mathcal{A}_p)$ condition*:

$$\sup_{I\subset\mathbb{R}} \left(\frac{1}{|I|}\int_I w(t)\,dt\right)\left(\frac{1}{|I|}\int_I w^{-1/(p-1)}(t)\,dt\right)^{p-1} < \infty, \tag{5}$$

where the supremum is taken over all finite intervals I in $\mathbb{R}$.

The function S is called the *generating function* of $\mathcal{Z}$. Non-negative functions w which enjoy the constraint (5) are said to belong to the class $(\mathcal{A}_p)$, see [18, Chapter V].

If $\mathcal{Z} = \{z_n\}$ is a complete interpolating sequence, each $F \in PW^p$ can be represented as

$$F(z) = \sum F(z_n)\frac{S(z)}{S'(z_n)(z-z_n)}. \tag{6}$$

This series converges in $L^p(\mathbb{R})$-norm and also

$$\|F\|_{PW_p} \asymp \|\{F(z_n)\}\|_{l^p}. \tag{7}$$

Here and in what follows the sign $\asymp$ means that the ratio of the two sides lies between two positive constants. In particular the system

$$\left\{\frac{S(z)}{S'(z_k)(z-z_k)}\right\}_k$$

is an unconditional basis in PW^p.

In what follows we always assume that the points from $\mathcal{Z}$ are enumerated so that $\operatorname{Re} z_k \leqslant \operatorname{Re} z_{k+1}$.

2.2. *Divided differences, $PW^{p,k}$, and $l^{p,k}(\mathcal{Z})$*

For any function f defined on $\mathcal{Z}$ the expression $f^{[k]}(z_n)$ denotes the forward divided difference of order k which is usually denoted by $[f(z_n), \dots, f(z_{n+k})]$, see [1, formulas (2.6.3) and (2.6.6)]. Such a difference is usually defined in terms of a real sequence $\{z_n, \dots, z_{n+k}\}$ but the same definition can be used in our case as well. The notation $f^{[k]}(z_n)$ is convenient since it allows us to use $f^{[k]}$ to succinctly denote the corresponding sequence of kth forward divided differences, namely $f^{[k]} = \{f^{[k]}(z_n)\}_{z_n\in\mathcal{Z}}$. Thus, for each k, $f^{[k]}$ may be viewed as a function on $\mathcal{Z}$ defined pointwise by

$$f^{[0]}(z_n) = f(z_n)$$

and, for $k = 1, 2, \dots,$

$$f^{[k]}(z_n) = \frac{f^{[k-1]}(z_{n+1}) - f^{[k-1]}(z_n)}{z_{n+k} - z_n}.$$

For every non-negative integer k we define $l^{p,k}(\mathcal{Z})$ to be the class of all functions $y : \mathcal{Z} \to \mathbb{C}$ whose corresponding sequence of kth divided forward differences, $y^{[k]}$, is in $l^p(\mathcal{Z})$. The linear space $l^{p,k}(\mathcal{Z})$ has a seminorm similar to that in $PW^{p,k}$:

$$\|f\|_{l^{p,k}(\mathcal{Z})} = \|\{f^{[k]}(z_n)\}\|_{l^p} \quad \text{for } f \in l^{p,k}(\mathcal{Z}).$$

2.3. The interpolation problem and its solution

We start with the elementary problem

For each n find such a function F_n in $PW^{p,k}$ that

$$F_n^{[k]}(z_m) = \begin{cases} 1 & \text{if } z_m = z_n, \\ 0 & \text{if } z_m \in \mathcal{Z} \setminus \{z_n\}. \end{cases} \tag{8}$$

Solution to (8) can be obtained directly. We introduce some additional notation in order to write it down. From the properties of complete interpolating sequences one can easily see that there exist a sequence of numbers $\{\alpha_n\} \subset \mathbb{R}$ and a sequence of contours $\{\gamma_n\}$ with the following properties:

- $0 < \delta_1 := \inf(\alpha_{n+1} - \alpha_n) < \sup(\alpha_{n+1} - \alpha_n) := \Delta_1 < \infty$;
- $\gamma_n \cap \{z : |\mathrm{Im}\, z| > 2H\} = \{\alpha_n + iy, |y| > 2H\}$;
- $\mathrm{length}(\gamma_n \cap \{z : |\mathrm{Im}\, z| \leqslant 2H\}) < K < \infty$;
- let γ_n split $\mathbb{C}$ into two domains $\Gamma_n^{\mp}$ located, respectively, to the left and right of γ_n. Then $\{z_k\}_{k \leqslant n} \subset \Gamma_n^-$ and $\{z_k\}_{k>n} \subset \Gamma_n^+$.

Let $p_k(z)$ be the polynomial defined by

$$p_k(z) = \begin{cases} 1 & \text{if } k = 1, \\ (z - z_{n+1})(z - z_{n+2}) \cdots (z - z_{n+k-1}) & \text{if } k = 2, 3, \ldots, \end{cases}$$

and consider the function

$$\Phi_n(z) = \frac{S(z)}{2\pi i} \int_{\gamma_n} \frac{p_n(\zeta)\, d\zeta}{(\zeta - z) S(\zeta)} + p_n(z) \chi_n^+(z). \tag{9}$$

Here $S(z)$ is the generating function of $\mathcal{Z}$ defined by (3) and $\chi_n^+(z)$ is the characteristic function of Γ_n^+. The properties of the generating function $S(z)$ imply that the integral is convergent and that $\Phi_n(z)$ is well defined as an analytic function for all z which are not on the contour γ_n. Furthermore, $\Phi_n(z)$ can be extended as an entire function of exponential-type no greater than π: it follows from the Sokhotskii–Plemelj formula that Φ_n is continuous in the whole plane and hence singularities along γ_n can be erased. The growth estimates are straightforward.

Theorem 1. *The entire function $\Phi_n(z)$ defined by* (9) *belongs to $PW^{p,k}$. Moreover*

$$\Phi_n^{[k]}(z_m) = 0, \quad m \neq n$$

and

$$\Phi_n^{[k]}(z_n) = z_{n+k} - z_n.$$

Thus the functions

$$\Psi_n = (z_{n+k} - z_n)^{-1}\Phi_n$$

solve the interpolation problems (8).

Remark. The expression (9) can be viewed as a solution of a specific $\bar{\partial}$ problem. The function $p_n\chi_n^+$ satisfies

$$-(p_n(z)\chi_n^+(z))^{[k]}(z_m) = (z_{n+k} - z_n)\delta_{n,m}$$

and is analytic for $z \notin \gamma_n$. The first term in the right-hand side of (9) is chosen so that

$$\bar{\partial}\left(\frac{S(z)}{2\pi i}\int_{\gamma_n}\frac{p_n(\zeta)\,d\zeta}{(\zeta - z)S(\zeta)}\right) = \bar{\partial}(p_n(z)\chi_n^+(z))$$

so the resulting function Φ_n belongs to $PW^{p,k}$ and solves the same interpolation problem.

The solution to the general problem (8) is reconstructed from the elementary solutions Ψ_n. Take polynomials $q_n(z)$ of degree at most $k-1$ and such that the functions

$$\Theta_n = \Psi_n - q_n \tag{10}$$

satisfy

$$\Theta_n^{(j)}(0) = 0, \quad j = 0, 1, \ldots, k-1. \tag{11}$$

These functions still solve the interpolation problem (8) and also $\Theta_n^{(k)} = \Psi_n^{(k)}$, $n \in \mathbb{Z}$.

Now take $\{\alpha(z_n)\} \in l^{p,k}(\mathcal{Z})$ and denote $a_n = \alpha^{[k]}(z_n)$.

Theorem 2. *For each sequence* $\{\alpha(z_n)\} \in l^{p,k}(\mathcal{Z})$ *the interpolation problem*

$$F(z_k) = \alpha(z_k), \quad k = 0, \pm 1. \pm 2, \ldots \tag{12}$$

has a solution in $PW^{p,k}$. *This solution is unique and*

$$\|F\|_{PW^{p,k}} \asymp \|\alpha\|_{l^{p,k}}. \tag{13}$$

This solution can be represented in the form

$$\Theta(z) = \sum_n a_n\Theta_n(z) + P(z), \tag{14}$$

where P is a polynomial of degree at most $k-1$. *The series* (14) *converges uniformly on each compact set in* $\mathbb{C}$ *and also in* $PW^{p,k}$.

3. Proofs of the main results: sine-type functions

In this section we prove that, if $\mathcal{Z}$ is a PW^p-complete interpolating sequence, then the mapping

$$T_{\mathcal{Z}} : F \mapsto F|_{\mathcal{Z}} \tag{15}$$

is a bounded operator from $PW^{p,k}$ into $l^{p,k}$ and has a zero kernel. Under the additional assumption that the generating function S of $\mathcal{Z}$ is a sine-type function (see definition below at Section 3.3) we prove Theorems 1 and 2 thus showing that this operator is invertible. We also provide explicit construction of the inverse operator. Later in Section 4 we develop techniques which are needed to study the general case i.e. when, for some $a \in \mathbb{R}$, $|S(x + ia)|^p \in (\mathcal{A}_p)$ and indicate the changes to be done in this case.

3.1. *Operator $T_{\mathcal{Z}}$*

First we mention that $\mathcal{Z}$ satisfies

$$\Delta = \sup |z_{n+1} - z_n| < \infty. \tag{16}$$

Indeed if this is not the case, one can pick a sequence $\{n_N\}$ with $|z_{n_N+1} - z_{n_N}| > N$, then the function $F_N = (z - (z_{n_N+1} - z_{n_N})/2)^{-1} \sin[\pi(z - (z_{n_N+1} - z_{n_N})/2)]$ delivers a contradiction to (7). Indeed, $\|F_N\|_{PW^p} \asymp 1$, while (easy estimate) $\|\{F_N(z_n)\}\|_{l^p} \to 0$ as $N \to \infty$.

Proposition. *Operator $T_{\mathcal{Z}}$ defined by* (15) *is a bounded operator from $PW^{p.k}$ into $l^{p,k}$ and* $\operatorname{Ker} T_{\mathcal{Z}} = 0$.

Proof. Without loss of generality one may assume that the sequence $\mathcal{Z}$ is located in the strip $\{z \in \mathbb{C}; 1 < \operatorname{Im} z < H\}$ for some $H > 0$. Denote by Π_n the rectangles whose boundary $\partial\Pi_n$ consists of the segments $\operatorname{Re} z_n - 1 + i[0, H]$, $[\operatorname{Re} z_n - 1, \operatorname{Re} z_{n+k} + 1]$, $\operatorname{Re} z_{n+k} + 1 + i[0, H]$, $[\operatorname{Re} z_n - 1, \operatorname{Re} z_{n+k} + 1] + iH$. That $T_{\mathcal{Z}}$ is bounded follows from the lemma below

Lemma 1. *For each n there exist functions $\omega_{n,j,l}$, $j = 0, 1, \ldots, k$, $l = 0, \ldots, k - j$ which are holomorphic in a neighborhood of Π_n and such that*

$$g^{[j]}(z_{n+l}) = \int_{\partial\Pi_n} g^{(j)}(\zeta)\omega_{n,j,l}(\zeta)\, d\zeta \tag{17}$$

for each function g which is holomorphic in Π_n and continuous in the closure of Π_n. In addition

$$|\omega_{n,j,l}(\zeta)| < M, \quad \zeta \in \partial\Pi_n, \tag{18}$$

where M does not depend upon n, j, and l.

Assume that this lemma is already proved. Let μ_n be the arc-length measure along the boundary $\partial\Pi_n$ and $\mu = \sum \mu_n$. It follows from (4) and (16) that μ is a Carleson measure in $\mathbb{C}^+$. (We refer the reader to [9] for definition and properties of the Carleson measures). Therefore for $F \in PW^{p,k}$:

$$\sum_n |F^{[k]}(z_n)|^p = \sum_n \left| \int_{\partial\Pi_n} F^{(k)}(\zeta)\omega_{n,k,0}(\zeta)\, d\zeta \right|^p$$

$$\leqslant \sum_n \int_{\partial\Pi_n} |F^{(k)}(\zeta)|^p |d\zeta| \left(\int_{\partial\Pi_n} |d\zeta|\right)^{p/q}$$
$$\leqslant \text{Const} \int_{\mathbb{C}^+} |F^{(k)}(\zeta)|^p |d\mu(\zeta)| \leqslant \text{Const}\|F\|_{PW^{p,k}}.$$

Here we use the fact that $F^{(k)}(z)e^{i\pi z} \in H^p(\mathbb{C}^+)$ and also that $|e^{i\pi z}| \asymp 1$ for $\operatorname{Im} z \in (0, H)$. $\square$

Proof of Lemma 1. We perform induction on j. For $j = 0$ the statement is evident, take $\omega_{n,0,l}(\zeta) = [2i\pi(\zeta - z_{n+l})]^{-1}$. Assuming that we have constructed all $\omega_{n,j-1,l}$ we mention that

$$g^{[j]}(z_{n+l}) = \frac{1}{z_{n+l+j} - z_{n+l}} \int_{\partial\Pi_n} g^{(j-1)}(\zeta)[\omega_{n,j-1,l+j}(\zeta) - \omega_{n,j-1,l}(\zeta)]\, d\zeta. \quad (19)$$

Fix some point $\zeta_n \in \partial\Pi_n$ and consider the function

$$\omega_{n,j,l}(\zeta) = \int_{\zeta_n}^{\zeta} [\omega_{n,j-1,l+j}(s) - \omega_{n,j-1,l}(s)]\, ds, \quad \zeta \in \partial\Pi_n,$$

where integration is performed along $\partial\Pi_n$. If

$$K := \int_{\partial\Pi_n} [\omega_{n,j-1,l+j}(s) - \omega_{n,j-1,l}(s)]\, ds = 0,$$

then $\omega_{n,j,l}$ may be extended as a function holomorphic in a vicinity of $\partial\Pi_n$ and relation (17) comes from integration by parts. If $K \neq 0$ integration by parts still gives

$$g^{[j]}(z_{n+l}) = \int_{\partial\Pi_n} g^{(j)}(\zeta)\omega_{n,j,l}(\zeta)\, d\zeta + Kg^{(j-1)}(\zeta_n),$$

valid for any function g holomorphic in Π_n. This yields $K = 0$ for if one replaces $g(z)$ by $g(z) + az^{j-1}$, the left-hand side of this relation remains unchanged whence the right-hand side increases by $aK(j-1)!$. This completes the proof of (17). Estimate (18) follows directly from the construction of the functions $\omega_{n,j,l}$. $\square$

3.2. Uniqueness

Lemma 2. *If $\mathcal{Z}$ is a complete interpolating sequence for PW^p then* $\ker T_{\mathcal{Z}} = \{0\}$.

Proof. Follows the standard pattern so we give just a short outline. Let, as before, S be the generating function for $\mathcal{Z}$ and $F \in \operatorname{Ker} T|_{\mathcal{Z}}$. Then $\Phi(z) = F(z)/S(z)$ is an entire function and it suffices to prove that $\Phi = 0$. Clearly Φ is a function of zero exponential type so it suffices to prove that $\Phi(iy) \to 0$ as $y \to \pm\infty$. Take an auxiliary function ϕ which is an entire function of exponential-type $\pi/2$, such that $\phi(t)t^{k+2} \to 0$ as $t \to \infty$, $t \in \mathbb{R}$ and in addition $\hat{\phi}(\xi) = 1$ at a vicinity of zero. Here as usual $\hat{\phi}$ stands for the Fourier transform. Let $F_1(x) = F * \phi(x)$ and $F_2 = F - F_1$. We have $F_1 \in PW^{p,k}$, in addition it has exponential type at most $\pi/2$. We also have $F_2^{(k)} = F^{(k)} - F_1^{(k)} \in PW^p$. The latter yields $F_2 \in PW^p$ since the Fourier transform of F_2 (if considered as a distribution) vanishes in a vicinity of zero, for such functions integration does not destroy the property of belonging to L^p. Now $F_1(iy)/S(iy) \to 0$ follows from the fact that type of F_1 is at most $\pi/2$. That $F_2(iy)/S(iy) \to 0$ is a standard estimate see e.g. [13]. This completes the proof of the lemma. $\square$

3.3. Sine-type functions

Let $\mathcal{Z}$ be a complete interpolating sequence for PW^p. Its generating function is called a *sine-type function* if it satisfies

$$|S(x + ia)| \asymp 1, x \in \mathbb{R},$$

for some $a \in \mathbb{R}$. This class of functions was introduced and studied in [10] (see also [11]). It is known that there is a constant H depending on S such that

$$|S(z)| \asymp e^{\pi|\operatorname{Im} z|}, \text{ whenever } |\operatorname{Im} z| > H. \tag{20}$$

We refer the reader to [11,20] for a detailed discussion of this class. In this section, we prove Theorems 1 and 2 under the additional assumption that S is a sine-type function. This restriction is stronger than (5). It allows us to show the main ideas by considering a simple case. The general case requires additional techniques which we think are of independent interest. These techniques are given in Section 4.

3.4. Proof of Theorem 1

First we prove that $\Psi_n \in PW^{p,k}$. It is clear that Ψ_n is an entire function of exponential-type π so it suffices to prove that $\Psi_n^{(k)} \in L^p(\mathbb{R})$, equivalently $\Phi_n^{(k)} \in L^p(\mathbb{R} + 2iH)$. For $z \notin \gamma_n$ we have

$$\Phi_n^{(k)}(z) = \frac{k!}{2\pi i} \sum_{j=0}^{k} \frac{1}{(k-j)!} \int_{\gamma_n} \frac{S^{(k-j)}(z) p_n(\zeta)\, d\zeta}{(\zeta - z)^{j+1} S(\zeta)}. \tag{21}$$

To see that $\Phi_n^{(k)} \in L^p(\mathbb{R} + 2iH)$ observe that

$$\int_{\gamma_n} \left\| \frac{S^{(k-j)}(\cdot) p_n(\zeta)}{(\zeta - \cdot)^{j+1} S(\zeta)} \right\|_{L^p((\mathbb{R}\setminus[\alpha_n - 1, \alpha_n + 1]) + 2iH)} |d\zeta| < \infty \tag{22}$$

since by the Bernstein theorem $S^{(k-j)}$ are bounded on $\mathbb{R} + 2iH$. The triangle inequality gives $\Phi_n^{(k)} \in L^p((\mathbb{R} \setminus [\alpha_n - 1, \alpha_n + 1]) + 2iH)$. The segment $[\alpha_n - 1, \alpha_n + 1] + 2iH$ is not essential since $\Psi_n^{(k)}$ is an entire function.

Now, for each m, we have

$$\Phi_n^{[k]}(z_m) = \frac{1}{z_{m+k} - z_m} \left((p_n \chi_n^+)^{[k-1]}(z_{m+1}) - (p_n \chi_n^+)^{[k-1]}(z_m) \right).$$

Noting that

$$(p_n\chi_n^+)^{[k-1]}(z_m)=\begin{cases}0, & m\leqslant n \text{ because } (p_n\chi_n^+)(z_j)=0,\ j\leqslant n+k-1;\\ 1, & m>n \text{ because } (p_n\chi_n^+)(z_j)=p(z_j),\ j>n,\end{cases}$$

we see that

$$\Phi_n(z)=(z_{n+k}-z_n)\Psi_n(z) \tag{23}$$

solves the problem (8). We also observe that

$$c_n:=z_{n+k}-z_n\asymp 1. \qquad \square \tag{24}$$

3.5. Convergence of the series of $\Psi_n^{(k)}$*'s*

Lemma 3. *Let* $\mathbf{a}=\{a_n\}\in l^p$ *be given. Then the series*

$$F_{\mathbf{a}}=\sum_{n\in\mathbb{Z}} a_n\Psi_n^{(k)} \tag{25}$$

is convergent in $L^p(\mathbb{R}+2iH)$, *and*

$$\|F_{\mathbf{a}}\|_{L^p(\mathbb{R}+2iH)}\leqslant \mathrm{Const}\|\mathbf{a}\|_{l^p}. \tag{26}$$

Proof. It suffices to prove (26) only for finite sequences $\mathbf{a}$, and then pass to the limit in the general case. Also it suffices to prove the convergence and obtain the corresponding estimate in the space $L^p(\mathbb{R}+2iH)$.

We need additional geometric constructions. Let $\alpha=\{\alpha_k\}$ and δ be the same as in the definition of the contours γ_n (see Section 2.3). Let also $E_1=\{\zeta\in\mathbb{R}+2iH, \operatorname{dist}(\zeta,\alpha+2iH)<\delta\}$, $E=(\mathbb{R}+2iH)\setminus E_1$.

It suffices to prove

$$\|F_{\mathbf{a}}\|_{L^p(E)}\leqslant \mathrm{Const}\|\mathbf{a}\|_{l^p}, \tag{27}$$

for we know already that $F_{\mathbf{a}}\in PW^p$ and also that $\|\cdot\|_{L^p(E_j)}\asymp\|\cdot\|_{L^p(\mathbb{R}+2iH)}$ for functions from PW^p, (see e.g. [11]). We have

$$\|F_{\mathbf{a}}\|_{L^p(E)}=\sup\left\{\left|\int_E F_{\mathbf{a}}(x)G(x)\,dx\right|;\ G\in L^q(E),\ \|G\|_{L^q(E)}\leqslant 1\right\}, \tag{28}$$

here $1/p+1/q=1$. Fix $G\in L^q(E)$, $\|G\|_{L^q(E)}\leqslant 1$. In view of (21) we obtain

$$\begin{aligned}\int_E F_{\mathbf{a}}(x)G(x)\,dx &= \sum_n a_n\int_E \Psi_n^{(k)}(x)G(x)\,dx\\ &= \frac{k!}{2\pi i}\sum_n a_n\int_E G(x)\sum_{j=0}^{k}\frac{1}{(k-j)!}\\ &\quad\times\int_{\gamma_n}\frac{S^{(k-j)}(x)p_n(\zeta)\,d\zeta}{(\zeta-x)^{j+1}S(\zeta)}\,dx\end{aligned}$$

$$= k! \sum_{j=0}^{k} \frac{1}{2\pi i} \frac{1}{(k-j)!}$$
$$\times \underbrace{\sum_n a_n \int_{\gamma_n} \int_E \left(\frac{G(x) S^{(k-j)}(x)\, dx}{(\zeta - x)^{j+1}} \right) \frac{p_n(\zeta)\, d\zeta}{S(\zeta)}}_{A_j}. \tag{29}$$

We will consider each A_j separately.

Denote now $l_n^{\pm} = \{\zeta \in \gamma_n : \pm(\operatorname{Im}\zeta - 2H) > 0\}$ and

$$A_j^{\pm} = \sum_n a_n \int_{l_n^{\pm}} \left(\int_E \frac{G(x) S^{(k-j)}(x)\, dx}{(\zeta - x)^{j+1}} \right) \frac{p_n(\zeta)\, d\zeta}{S(\zeta)}. \tag{30}$$

For the sake of simplicity we consider A_j^+'s only, A_j^-'s can be estimated in a similar way. Consider the contours [2]

$$\kappa = \{\zeta = \xi + i\eta : |\xi| < \delta, \eta = 2(\xi^2 - \delta^2)^2\}, \quad \lambda = E \cup (\cup_n(-\kappa + \alpha_n + 2iH)).$$

The curve λ splits $\mathbb{C}$ into two parts $\mathbb{C}_\lambda^{\pm}$, we have $l_n^+ \subset \mathbb{C}_\lambda^+$, and $\operatorname{dist}(l_n^+, \lambda) > 2\delta^4$, also $\operatorname{dist}(l_n^+, \lambda + i\delta^4) > \delta^4$.

Let $\omega : \mathbb{C}^+ \to \mathbb{C}_\lambda^+$ be the conformal mapping normalized by $\omega(i\eta) = i\eta + O(1),\ \eta \to +\infty$. (For existence of such mapping see [3]). Denote the space $H^q(\mathbb{C}_\lambda^+)$ as

$$H^q(\mathbb{C}_\lambda^+) = \{f(z),\ f \text{ analytic in } \mathbb{C}_\lambda^+,\ f(\omega(\cdot)) \in H^q(\mathbb{C}^+)\}. \tag{31}$$

Similarly one can define the Hardy space $H^2(\mathbb{C}_\lambda^+ + i\delta^4)$ in the shifted domain $\mathbb{C}_\lambda^+ + i\delta^4$. Denote now

$$\Omega_j(\zeta) = \int_E \frac{G(x) S^{(k-j)}(x)\, dx}{(\zeta - x)^{j+1}} = \frac{1}{j!} \frac{d^j}{d\zeta^j} \tilde{\Omega}_j(\zeta),$$
$$\tilde{\Omega}_j(\zeta) = \int_E \frac{G(x) S^{(k-j)}(x)}{\zeta - x}\, dx. \tag{32}$$

By the Bernstein theorem, $\sup_{z \in (\mathbb{R}+2iH)} |S^{(k-j)}(z)| < \infty$, so

$$\tilde{\Omega}_j \in H^q(\mathbb{C}_\lambda^+) \text{ and } \|\tilde{\Omega}_j\|_{H^q(\mathbb{C}_\lambda^+)} \leqslant \text{Const} \|G\|_{L^q(E)}. \tag{33}$$

Therefore Ω_j belongs to the Hardy space in the inner domain: $H^q(\mathbb{C}_\lambda^+ + i\delta^4)$:

$$\Omega_j \in H^q(\mathbb{C}_\lambda^+ + i\delta^4) \text{ and} \|\Omega_j\|_{H^q(\mathbb{C}_\lambda^+ + i\delta^4)} \leqslant \text{Const} \|G\|_{L^q(E)}. \tag{34}$$

Let now $L = \cup l_n^+$. Define a measure μ on L so that its restriction to each l_n is just $(|p_n(\zeta)|)/(|S(\zeta)|)|d\zeta|$. Invoking the standard distortion theorems for conformal mappings

[2] Actually by λ one can take an arbitrary smooth contour which meets some natural restriction, contains E, and separates the sets l_n's from the lower half-plane. It is just for convenience we define these contours explicitly.

(see e.g. [3,19]) and also the estimate of S from below (see (20)) one can easily see that μ is a "Carleson measure for the domain $\mathbb{C}^+_\Gamma + i\delta^2$". This yields

$$\left(\sum_n \int_{l_n} |\Omega_j(\zeta)|^q \frac{p_n(\zeta)\, d\zeta}{|S(\zeta)|}\right)^{1/q} \leqslant \mathrm{Const}\|\Omega_j\|_{H^q(\mathbb{C}^+_\lambda)}. \tag{35}$$

Now, by the Hölder inequality,

$$\begin{aligned} |A_j| &\leqslant \mathrm{Const}\left(\sum |a_n|^p\right)^{1/p}\left(\sum_n\left(\int_{l_n}|\Omega_j(\zeta)|\left|\frac{p_n(\zeta)}{S(\zeta)}\right||d\zeta|\right)^q\right)^{1/q}\\ &\leqslant \mathrm{Const}\left(\sum |a_n|^p\right)^{1/p}\left(\sum_n\int_{l_n}|\Omega_j(\zeta)|^q\left|\frac{p_n(\zeta)}{S(\zeta)}\right||d\zeta|\right.\\ &\quad\left.\times\left(\int_{l_n}\left|\frac{p_n(\zeta)}{S(\zeta)}\right||d\zeta|\right)^{p/q}\right)^{1/q}\\ &\leqslant \mathrm{Const}\left(\sum |a_n|^p\right)^{1/p}\left(\sum_n\int_{l_n}|\Omega_j(\zeta)|^q\left|\frac{p_n(\zeta)}{S(\zeta)}\right||d\zeta|\right)^{1/q}\\ &\leqslant \mathrm{Const}\left(\sum |a_n|^p\right)^{1/p}\|\Omega_j\|_{H^q(\mathbb{C}^+_{\lambda+i\delta^4})}, \end{aligned} \tag{36}$$

the latter inequality follows from the fact that μ is a Carleson measure for $\mathbb{C}^+_{\lambda+i\delta^4}$. Now it remains to use (33) and (34). The quantities A_j^- can be estimated similarly. This completes the estimate of $\|F_{\mathbf{a}}\|_{L^p(E_p)}$.

Let polynomials q_n of degree at most $k-1$ and functions Θ_n be defined by relations (10), (11). The functions Θ_n still solve the interpolation problem (8) and also $\Theta_n^{(k)} = \Phi_n^{(k)}$, $n \in \mathbb{Z}$.

This allows us to consider the original interpolation problem (12). Take $\{\alpha(z_n)\} \in l^{p,k}(\mathcal{Z}))$ and denote $a_n = \alpha^{[k]}(z_n)$. We have $\mathbf{a} = \{a_n\} \in l^p$, hence the series

$$G_{\mathbf{a}} = \sum_{n\in\mathbb{Z}} a_n \Theta_n^{(k)} \tag{37}$$

is convergent in $L^p(\mathbb{R})$ and (since all $\Theta_n^{(k)}$ are entire functions of exponential type at most π) on each compact set in $\mathbb{C}$. It also follows from (26) that, for each compact set $K \subset \mathbb{C}$,

$$\left|\sum_{n\in\mathbb{Z}} a_n \Theta_n^{(k)}(z)\right| \leqslant C(K)\|\alpha\|_{l^{p,k}(\mathcal{Z})}, \quad z \in K, \tag{38}$$

here $C(K)$ depends upon K only. Since $\Theta_n^{(k-1)}(0) = 0$ this implies the compactwise convergence of the series

$$\sum_{n\in\mathbb{Z}} a_n \Theta_n^{(k-1)} \tag{39}$$

with an estimate similar to (38). Repeating this reasoning k times we see that the series

$$G(z) = \sum_{n\in\mathbb{Z}} a_n \Theta_n(z) \tag{40}$$

is uniformly convergent on each compact set in $\mathbb{C}$ and

$$|G(z)| \leqslant C(K)\|\alpha\|_{l^{p,k}(\mathcal{Z})}, \quad z \in K \tag{41}$$

(here the constant $C(K)$ is different from that in (38), generally speaking). In addition we have

$$G^{[k]}(z_n) = \alpha^{[k]}(z_n), \quad n \in \mathbb{Z}. \tag{42}$$

A solution to the interpolation problem (8) can now be obtained in the form

$$F(z) = G(z) + P(z), \tag{43}$$

where $P(z)$ is a polynomial of degree at most $k-1$ chosen so that $F(z_n) = \alpha(z_n)$, $n = 0, 1, \ldots, k-1$. Directly from the construction we see that

$$F \in PW^{p,k} \quad \text{and} \quad \|F\|_{PW^{p,k}} \leqslant \text{Const}\|\alpha\|_{l^{p,k}(\mathcal{Z})}. \tag{44}$$

This completes the proof of Theorem 2 in the case when the generating function $S(z)$ is a sine-type function. $\square$

4. General complete interpolating sequences

In the general case the generating function S need not be a sine-type function. In this case we need auxiliary (mainly) technical tools to complete the proof. In this section, we collect these tools and indicate the adjustments to be done in the proof of the model case.

4.1. Weighted Hardy spaces

Recall the definition of outer functions in $\mathbb{C}^+$ (see e.g. [2,9] for the details).

A function $h(z)$ holomorphic in $\mathbb{C}^+$ is called an outer function in $\mathbb{C}^+$ if for almost all $t \in \mathbb{R}$, h has non-tangential boundary values $h(t) = h(t + i0)$ satisfying

$$\int_{-\infty}^{\infty} \frac{\log|h(t)|}{1+t^2}\, dt < \infty \tag{45}$$

and which can be expressed in the form

$$h(z) = \exp\left\{\int_{-\infty}^{\infty} \log|h(t)| \left\{\frac{1}{t-z} - \frac{t}{t^2+1}\right\} dt\right\}, \quad z \in \mathbb{C}^+. \tag{46}$$

To each weight function $w(t) \geqslant 0$, $t \in \mathbb{R}$, which satisfies

$$\int_{-\infty}^{\infty} \frac{\log w(t)}{1+t^2}\, dt < \infty, \tag{47}$$

there corresponds an outer function $h(z) = h_w^+(z),\ z \in \mathbb{C}^+$ such that $|h(t)| = w(t), t \in \mathbb{R}$. This function is defined by (46). Similar definitions can be done for shifted half-planes $\mathbb{C}_a^\pm = \{z : \pm\Im(z-a) > 0\}$, $a \in \mathbb{R}$. In what follows we consider only weight functions satisfying (47).

Given a weight function w consider the weighted space

$$L_w^p(\mathbb{R}) = \{\phi(t),\ t \in \mathbb{R};\ \|\phi\|_{L_w^p}^p := \int_{-\infty}^{\infty} |\phi(t)|^p w(t)\, dt < \infty\}. \tag{48}$$

By weighted Hardy space $H_w^p(\mathbb{C}^+)$ we mean

$$H_w^p(\mathbb{C}^+) = \frac{1}{(h_w^+)^{1/p}} H^p(\mathbb{C}^+), \tag{49}$$

here $H^p(\mathbb{C}^+)$ is the classical Hardy space. For functions $f \in H_w^p(\mathbb{C}^+)$ we have

$$\|f\|_{H_w^p(\mathbb{C}^+)}^p = \sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^p |h(x+iy)|\, dx = \|f\|_{L_w^p}^p. \tag{50}$$

Functions $w \in (\mathcal{A}_p)$ satisfy

$$\int_{-\infty}^{\infty} \frac{w(t)}{1+|t|^p}\, dt < \infty. \tag{51}$$

This follows from the fact that the Hilbert transform of the characteristic function of the interval $(0, 1)$, decays as $(1+|t|)^{-1}$ and belongs to $L_w^p(\mathbb{R})$. Also we have $w^* := w^{-1/(p-1)} \in (\mathcal{A}_q)$ with $q = p/(p-1)$. Therefore

$$\int_{-\infty}^{\infty} \frac{w(t)^{-1/(p-1)}}{1+|t|^q}\, dt < \infty.$$

The last two relations yield (47). So, for $w \in (\mathcal{A}_p)$, the corresponding outer functions as well as weighted Hardy spaces are always well-defined.

Moreover, when $w \in (\mathcal{A}_p)$, the Hilbert transform is a bounded operator in $L_w^p(\mathbb{R})$, (see e.g. [8]) so each $\phi \in L_w^p(\mathbb{R})$ can be split as

$$\phi = f_\phi^+ + f_\phi^-;\ f_\phi^\pm \in H^p(\mathbb{C}^\pm);\ \|\phi\|_{L_w^p} \asymp \|f_\phi^+\|_{L_w^p} + \|f_\phi^-\|_{L_w^p}, \tag{52}$$

with $f_\phi^\pm$ defined as

$$f_\phi^\pm(z) = \frac{1}{2i\pi} \int_{\mathbb{R}} \frac{\phi(t)}{t-z}\, dt,\ z \in \mathbb{C}^\pm. \tag{53}$$

This implies in particular that the dual space $H_w^p(\mathbb{C}^+)^*$ can be realized as $H_{w^*}^q(\mathbb{C}^-)$. For each $g \in H_{w^*}^q(\mathbb{C}^-)$ the corresponding functional $\Phi_g \in H_w^p(\mathbb{C}^+)^*$ is defined as

$$\Phi_g(f) = \int_{-\infty}^{\infty} f(t)g(t)\, dt, \quad \|\Phi_g\|_{H_w^p(\mathbb{C}^+)^*} \asymp \|g\|_{H_{w^*}^q(\mathbb{C}^-)}.$$

4.2. Entire functions satisfying $(\mathcal{A}_p)$ condition

We start with the inner–outer factorization of S in the upper and lower half-planes, see for example [7] for details.

Given $a \in \mathbb{R}$ we denote $\mathbb{C}_a^{\pm} = \{z \in \mathbb{C}; \pm(\Im z - a) > 0\}$. Assuming S to be fixed denote by $h_a^{\pm}(z)$ the outer functions in $\mathbb{C}_a^{\pm}$ satisfying

$$|h_a^{\pm}(x + ia)| = |S(x + ia)|, \quad x \in \mathbb{R}.$$

Then

$$S(z) = e^{\mp\pi(z-ia)} h_a^{\pm}(z) B_a^{\pm}(z), \quad z \in \mathbb{C}_a^{\pm}, \tag{54}$$

where $B_a^{\pm}(z)$ is the Blaschke product in $\mathbb{C}_a^{\pm}$ corresponding to the zeros of S which fall in $\mathbb{C}_a^{\pm}$. If S does not vanish in $\mathbb{C}_a^{\pm}$, we put $B_a^{\pm}(z) = 1$.

The following statement seems also to be of independent interest.

Theorem 3. *Let $\mathcal{Z} = \{z_k\}$ be a complete interpolating sequence for PW^p and S be its generating function. Denote*

$$\rho = 2 \sup_k |\mathrm{Im}\, z_k|. \tag{55}$$

For each integer $l > 0$ there exists $a > 0$ such that

$$\left| \frac{S^{(j)}(z)}{S(z)} \right| \asymp 1, \text{ whenever } |\mathrm{Im}\, z| > a, \; j = 1, \ldots, l. \tag{56}$$

Proof. First we recall that if $p = 2$ condition $w \in (\mathcal{A}_p)$ is equivalent to the Helson–Szegö condition (see e.g. [2])

$$w = \exp(u + \tilde{v}), \quad u, v \in L^{\infty}(\mathbb{R}), \; \|v\|_{\infty} < \pi/4. \tag{57}$$

Here $\tilde{v}$ stays for the Hilbert transform of the function v. It follows immediately from (5) that $w^{-1/(p-1)} \in (\mathcal{A}_q)$. Also for $p \geqslant 2$, $w \in (\mathcal{A}_p)$ yields $w \in (\mathcal{A}_2)$. Combining this fact we see that, for each p, $1 < p < \infty$ functions $w \in (\mathcal{A}_p)$ admit the representation

$$w = \exp(u + \tilde{v}), \quad u, v \in L^{\infty}(\mathbb{R}). \tag{58}$$

The statement of the theorem follows from this representation. Assume for simplicity that $j = 1$, the rest can be proved by induction.

We mention that $\rho < \infty$, otherwise $\mathcal{Z}$ evidently cannot be a complete interpolating sequence. It follows from (54) that

$$\frac{S'(z)}{S(z)} = -\pi i + \frac{h_\rho^{+\prime}(z)}{h_\rho^{+}(z)}, \quad \Im z > \rho,$$

so it suffices to prove

$$\frac{h_\rho^{+\prime}(z)}{h_\rho^{+}(z)} \to 0 \text{ as } \Im z \to +\infty. \tag{59}$$

To this end we recall that $|h_\rho^+(x+i\rho)|^p = |S(x+i\rho)|^p \in (\mathcal{A}_p)$, so

$$\log|h_\rho^+(x+i\rho)| = u(x) + \tilde{v}(x), \text{ for some } u, v \in L^\infty.$$

Now (46) yields

$$\frac{h_\rho^{+\prime}(z)}{h_\rho^+(z)} = \int_{-\infty}^{\infty} \frac{u(t)}{(z-i\rho-t)^2}\,dt + \int_{-\infty}^{\infty} \frac{\tilde{v}(t)}{(z-i\rho-t)^2}\,dt = a_1(z) + a_2(z).$$

That $a_1(z) \to 0$ as $\Im z \to +\infty$ is straightforward. In order to prove the same statement for a_2 it suffice to observe that

$$\int_{-\infty}^{\infty} \frac{\tilde{v}(t)}{(z-i\rho-t)^2}\,dt = \int_{-\infty}^{\infty} \frac{v(t)}{(z-i\rho-t)^2}\,dt.$$

This completes the proof of (56) for $j=1$. $\square$

Remarks. 1. It follows from Theorem 3 that, in the case when the zero set of the derivative of generating function is uniformly discrete, this set also is a complete interpolating sequence for $PW^{p,k}$. In the non-discrete case one can use divided differences and the block summation procedure in order to study the corresponding interpolation problem. We refer the reader to [10] for an explicit description of this procedure.

2. The statement of Theorem 3 can be obtained from a more general result in [4] which relates to entire functions of completely regular growth. The proof we give in our case is much simpler and direct than one in [4].

4.3. The general case

Now we indicate the changes to be done in the proofs of Theorems 1 and 2 to obtain the general case, i.e. when $|S(x+ih)|^p \in (\mathcal{A}_p)$ for some $h \neq 0$. The proof of Theorem 1 goes along the same line if one takes relation (51) into account. Therefore we concentrate on Theorem 2 only, keeping the same notation as in Section 3. We have to estimate $\|F_{\mathbf{a}}\|_{L^p(\mathbb{R}+iH)}$. As well as in the model case we shall replace this norm by an equivalent norm $\|\cdot\|_{L^p(E)}$. Now it is more convenient to consider the norm $\|F_{\mathbf{a}}\|_{L^2(\mathbb{R}+iH)}$, this is an equivalent norm.

As in the case of sine-type functions, (27) would follow from

$$\sum_n \int_{l_n^\pm} \left|\Omega_j(\zeta)\right|^p \left|\frac{p_n(\zeta)}{S(\zeta)}\,d\zeta\right| \leqslant \text{Const}, \qquad j = 1, \ldots, k, \tag{60}$$

where as before Ω_j are defined by (33) and $G \in L^q(E)$, $\|G\|_{L^q(E)} \leqslant 1$.

The function $|S(x+2iH)|$ is no longer bounded from above and from below so we have to introduce the corresponding *weighted* Hardy spaces. Without loss of generality we may assume that (56) holds for $z \in \lambda$.

Define the weight function $w(\zeta) = |S(\zeta)|^p$, $\zeta \in \Gamma_1$ and consider the corresponding weighted Hardy space in $\mathbb{C}_\lambda^+$

$$H_w^p(\mathbb{C}_\lambda^+) = h_\lambda^+ H^p(\mathbb{C}_\lambda^+),$$

here h_λ^+ is the outer function in $\mathbb{C}_\lambda^+$ satisfying $|h_\lambda^+(\zeta)| = |S(\zeta)|$, $\zeta \in \lambda$. Relation (54) now yields

$$|h_+(z)| \asymp |h_\lambda(z)|, \quad z \in \mathbb{C}_\lambda^+. \tag{61}$$

It follows now from (56) that the functions $\phi_j(\zeta) = G(\zeta)S^{(k-j)}(\zeta)$ satisfy $\phi_j(\zeta)/S(\zeta) \in L^q(E)$. Since $|S(t+2iH)|^p \in (\mathcal{A}_p)$ we have

$$\tilde{\Omega}_j \in H^2(\mathbb{C}_\lambda^+; |S(\zeta|^q), \quad \|\tilde{\Omega}_j\|_{H^q(\mathbb{C}_\lambda^+;|S(\zeta|^q)} \leqslant \text{Const}.$$

Taking (61) into account one can rewrite the latter inequality as

$$\|h_+\tilde{\Omega}_j\|_{H^q(\mathbb{C}_\lambda^+)} \leqslant \text{Const}.$$

Now applying (61) once again we obtain

$$h_+\Omega_j \in H^q(\mathbb{C}_\lambda^+ + i\delta^4), \quad \|h_+\Omega_j\|_{H^2(\mathbb{C}_\lambda^+)} \leqslant \text{Const}.$$

Define by μ_n the restriction of the measure $e^{-\pi\Im z}|p_n(z)||\,dz|$ to the ray l_n^+. It remains now to mention that $\sum \mu_n$ is a Carleson measure in $\mathbb{C}_{\Gamma_1}^+ + i\delta/2$.

The proof of (60) for l_n^- goes along the same lines, the major difference relates to the fact that the Blaschke product B appears when one writes inner–outer factorization for S in the domain $\mathbb{C}_{\Gamma_1}^-$. This, however does not lead one to any additional complication, because it follows from the Carleson condition for $\mathcal{Z}$ that $|B|$ is bounded from below and from above in a neighborhood of union of rays l_n^-. This completes the proof of Theorem 1.

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